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Automata and Zappa-Szép products of groups

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Zappa-Szép products

Let $H, K \leq G$.

- If $G = HK$ with $H \cap K = 1$ and $H, K \lhd G$ then G is an *internal direct product* of H and K, $G = H \times K$.
- If $G = HK$ with $H \cap K = 1$ and $H \lhd G$ then G is an *internal semi-direct product* of H and K, $G = H \rtimes K$.
- If $G = HK$ with $H \cap K = 1$ then G is an *internal Zappa-Szép ~~direct~~ product* of H and K, $G = H \bowtie K$.

Other names: general product, knit product, exact factorization.

These structures are named after

- Guido Zappa (1940, in Italian), and
- Jenö Szép (1948)

who independently studied these products in the mid-20th century.

Other folk deserve a mention though:

- de Séguier (1904, in French)
- Miller (1935)
- Neumann (1935)

American Mathematical Society MathSciNet Mathematical Reviews

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Publications results for "MR Number=(45719)"

MR0045719 (13,621l) Reviewed

Douglas, Jesse

On finite groups with two independent generators. IV. Conjugate substitutions.

Proc. Nat. Acad. Sci. U.S.A. 37 (1951), 808–813.

20.0X

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Let a group G of order mn possess cyclic subgroups A and B, of orders m and n, respectively, and let the intersection of A and B consist of the unit-element only. If A and B are generated by a and b, respectively, then from the fact that A and B must be permutable it follows that there exist formulae $ba^x = a^{\theta(x)}b^{\psi(x)}$ and $b^y a = a^{\omega(y)}b^{\varphi(y)}$. Here $\theta(x)$ and $\varphi(y)$ are permutations of the residue classes modulo m and n, respectively. The idea of the author is to attack directly this permutational aspect of the situation. He shows that $\theta(x)$ and $\varphi(y)$ determine the structure of the group G completely; indeed $b^y a^x = a^{\theta^y(x)}b^{\varphi^x(y)}$ and $a^u b^y a^x b^v = a^{u+\theta^y(x)}b^{v+\varphi^x(y)}$. He determines properties of permutations which are necessary and sufficient to associate them with a group G in the manner indicated above. With the help of these criteria he finds many properties of groups which are the product of two permutable cyclic subgroups, e.g. that they are soluble. He determines the normalizers of A and B, respectively, also maximal Abelian subgroups and the order of the centre of G. The discussion is in permutational rather than group-theoretical language throughout. The extensive literature on permutable products of two cyclic groups is not taken into account. For a survey see, e.g., the paper of Wielandt reviewed above.

Reviewed by K. A. Hirsch

The external view

If $kh \in G$ with $k \in K$ and $h \in H$, we can reverse this product: there exists unique elements $k \cdot h \in K$ and $\theta_k(h) \in H$ such that $kh = \theta_k(h)(k \cdot h)$.

if $kh \in G$ $\exists\, k' \in K, h' \in H$ s.t.

there exists unique elements $k \cdot h \in K$ and $\theta_k(h) \in H$ such that $kh = \theta_k(h)(k \cdot h)$.

As $G = HK$, if $kh \in G$, $\exists\, k' \in K, h' \in H$ s.t.
$kh = h'k'$.

h' depends on h & k $\rightsquigarrow \theta_k(h)$

k' depends on h & h $\rightsquigarrow k \cdot h$

We therefore have two functions

$$(k, h) \longmapsto \theta_k(h) \in H, \qquad (k, h) \longmapsto k \cdot h \in K$$

called *the mutual actions defined by the multiplication.*

$kh = \theta_k(h)(k\cdot h)$ $\quad \in K$

$$k h_1 h_2 = \theta_k(h_1)(k\cdot h_1) h_2$$

$$= \underbrace{\theta_k(h_1)\,\theta_{k\cdot h_1}(h_2)}_{\in H} \underbrace{[(k\cdot h_1)\cdot h_2]}_{K}$$

$$= \theta_k(h_1h_2)(k\cdot h_1h_2).$$

$$\theta_k(h_1h_2) = \theta_k(h_1)\,\theta_{k\cdot h_1}(h_2)$$

& $(k\cdot h_1)\cdot h_2 = k\cdot(h_1h_2)$ $\qquad k_1k_2h$

This general idea allows for generalisations to other algebraic structures (see Brin *On the Zappa–Szép product*, Comm. Algebra 2005).

Take these conditions as axioms which two functions must satisfy:

Let H, K be semigroups where $h, h_1, h_2 \in H$ and $k, k_1, k_2 \in K$:

$$
\begin{aligned}
k \cdot (h_1 h_2) &= (k \cdot h_1) \cdot h_2, \\
\theta_k(h_1 h_2) &= \theta_k(h_1)\theta_{k \cdot h_1}(h_2), \\
\theta_{k_1 k_2}(h) &= \theta_{k_1}(\theta_{k_2}(h)), \\
(k_1 k_2) \cdot h &= (k_1 \cdot \theta_{k_2}(h))(k_2 \cdot h)
\end{aligned}
$$

$h_1 k_1 h_2 k_2 = h_1 \theta_{k_1}(h_2)(k_1 \cdot h_2) k_2$

Then under the multiplication

$$(h_1, k_1)(h_2, k_2) = (h_1 \theta_{k_1}(h_2), (k_1 \cdot h_2)k_2),$$

the set $H \times K$ defines a semigroup called the *external Zappa–Szép semigroup of H and K*, denoted $H \bowtie K$.

If H and K are monoids and the multiplication satisfies

$$
\begin{aligned}
k \cdot 1_H &= k \\
\theta_{1_K}(h) &= h
\end{aligned}
$$

action

$\theta_k(1_H) = 1_H$

then $H \bowtie K$ is a monoid, called the *external Zappa–Szép monoid of H and K*.

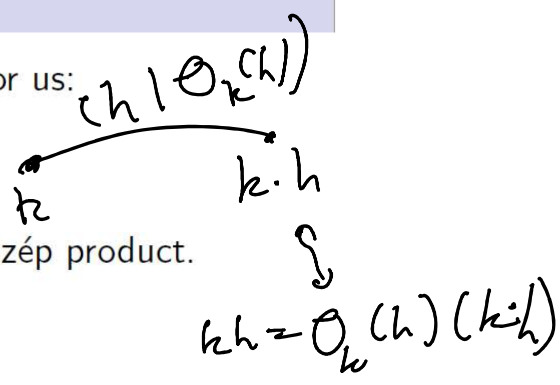
If H and K are groups then the external Zappa–Szép monoid is in fact a group, the *external Zappa–Szép group of H and K*.

1983 Kunze

Automata

An automaton $\mathcal{A}_{H \bowtie K}$ is a labeled, directed graph. For us:

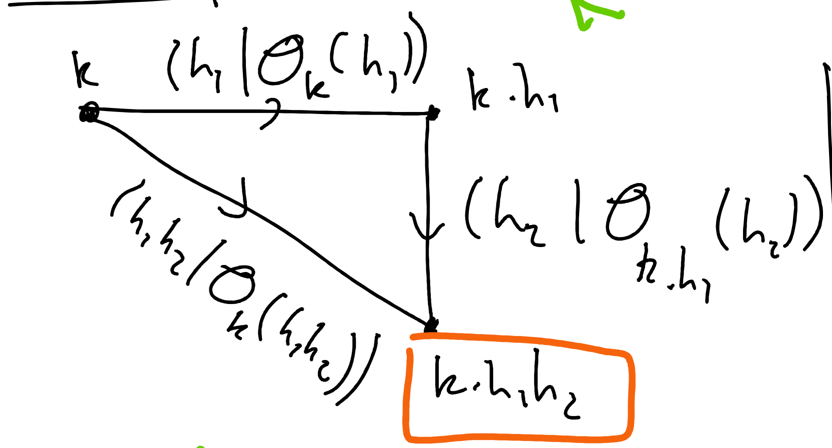
- vertices from (a subset of) K,
- edge labels from (a subset of) $H \times H$.



Want $\mathcal{A}_{H \bowtie K}$ to encode the structure of the Zappa–Szép product.

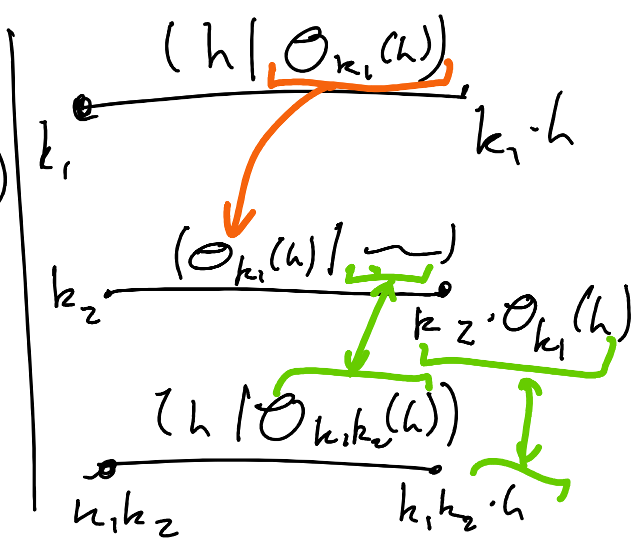
$$
\begin{aligned}
k \cdot (h_1 h_2) &= (k \cdot h_1) \cdot h_2, \\
\theta_k(h_1 h_2) &= \theta_k(h_1)\theta_{k \cdot h_1}(h_2), \\
\theta_{k_1 k_2}(h) &= \theta_{k_1}(\theta_{k_2}(h)), \\
(k_1 k_2) \cdot h &= (k_1 \cdot \theta_{k_2}(h))(k_2 \cdot h)
\end{aligned}
$$

Serial processing



$$\theta_k(h_1)\theta_{k \cdot h_1}(h_2) = \theta_k(h_1 h_2)$$

Sequential processing



Presentations

If H and K are groups then $H \bowtie K$ has relative presentation

$$\langle H, K \mid T_{H \bowtie K} \rangle$$

where $T_{H \bowtie K}$ consists of elements of the form $kh = \theta_k(h)(k \cdot h)$.

Question

Given H, K and $T_{H \bowtie K}$ as above, under what conditions is

$$\langle H, K \mid H, K, T_{H \bowtie K} \rangle$$

a Zappa–Szép semigroup/monoid/group $H \bowtie K$?

$\langle \underbrace{a, b}_{H}, \underbrace{x, y}_{K} \mid ax = ya, \; ay = ya, \; \_\_\_ \rangle \Rightarrow x = y$

Plan of action:

1. Fix an alphabet X.
2. Fix a group H.
3. Define $T_{H \bowtie K}$ via an automaton $\mathcal{A}_{(H,X)}$.
4. Placing restrictions on $\mathcal{A}_{(H,X)}$ gives:
   - a semigroup $H \bowtie X^+$. $K = X^+$
   - a monoid $H \bowtie X^*$. $K = X^*$
   - a group $H \bowtie F(X)$. $K = F(X)$

Theorem

Suppose $\mathcal{A}_{(H,X)}$ satisfies the serial processing condition. Then

$$\mathrm{Sem}[H, X \mid xh = \theta_x(h)(x \cdot h) \; (h \in H, x \in X)]$$

$= \rho(h, \theta_x(h))$

x $x \cdot h$

is a Zappa-Szép semigroup of H and X^+.

1 *Proof.* The rewriting system defined by

$xg \longrightarrow \theta_x(g)(x \cdot g) \quad x \in X$

is complete $\longrightarrow \forall \, w \in X^+,$

$$\text{is complete} \longrightarrow \forall\, w \in X^*,$$
$$(\exists!\, g' \in H \;\&\; w' \in X^* \text{ s.t. } wg = g'w'.$$

$$wg = w_1 x_1 g$$
$$= w_1 \theta_{x_1}(g)(x_1 \cdot g)$$
$$= \dots$$

2 So the map $\varphi \longrightarrow H \bowtie X^*$ is injective & a homomorphism

$$x \longmapsto (1, x)$$
$$g \longmapsto (g, \varepsilon)$$

$\boxtimes$

Additional conditions

Theorem

Suppose $\mathcal{A}_{(H,X)}$ satisfies the serial processing condition, and both $\mathcal{A}_{(H,X)}$ and the dual automaton $\mathcal{A}^{\mathbf{d}}_{(H,X)}$ are invertible. Then

$$\mathrm{Mon}[H, X \mid xh = \theta_x(h)(x \cdot h)\ (h \in H, x \in X)]$$

is a Zappa–Szép monoid of H and X^*, and

$$\mathrm{Grp}[H, X \mid xh = \theta_x(h)(x \cdot h)\ (h \in H, x \in X)]$$

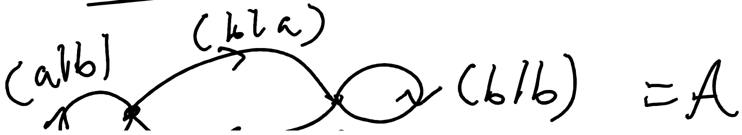
is a Zappa–Szép group of H and $F(X)$.

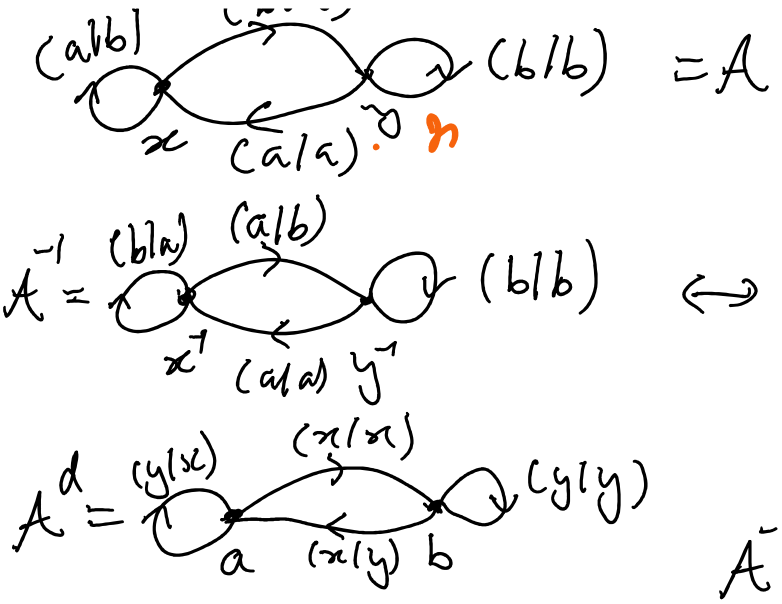
Dual and inverse automata

$F(a,b) \bowtie F(x,y)$: $Gp\langle a, b, x, y, z \mid$

non-example:

$xa = bx$
$xb = ay$
$ya = ax$





$xb = ay$

$ya = ax$

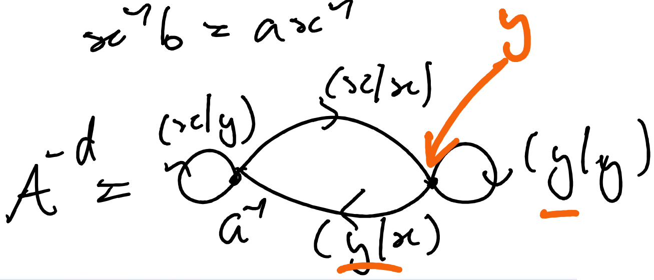
$yb = bx$).

$\longleftrightarrow$ $xa = bx$

$ax^{-1} = x^{-1}b$

$x^{-1}b = ax^{-1}$

NO



Application: Automorphism-induced HNN-extensions

Theorem

Fix G a group.

- *Let $H \leq G$ be a subgroup.*
- *Let T be a transversal for G/H.*
- *Let $\phi \in \mathrm{Aut}(G)$ be an automorphism.*

Then the automorphism-induced HNN-extension

$$\langle G, t \mid tht^{-1} = \phi(h)\ (h \in H)\rangle$$

is isomorphic to the Zappa–Szép product

$$G \bowtie F(X) = \langle G, X \mid x_\tau g = \phi(g) x_{\overline{\tau g}}\ (\tau \in T, g \in G)\rangle.$$

Conclusions

- There is a link between Zappa–Szép products and automata for semigroups.
- This can be generalised and exploited to answer the question "when do a set of relators of the form $zx = \theta_z(x)(z \cdot x)$ give a Zappa–Szép product?"

Thank you for your attention!